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ON THE STATISTICAL THEORY OF VISCOELASTIC PROPERTIES OF ASYMMETRIC MEDIA

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Within the scope of the Kubo linear reaction based on the classical Gibbs formalism without involving known additional representations, expressions in terms of the time correlation functions are obtained for four tensors of the viscosity coefficients of an asymmetric medium. Independently of the time correlation function apparatus, expressions are established for the ultimate high-frequency and adiabatic elastic moduli by analyzing the increments of the stress tensors upon application of a small strain.

Macroscopic phenomena of the internal (rotational) degrees of freedom are the center of attention of phenomenological theories of asymmetric media (see [1–4], for example). According to these latter, the motion of a continuum is described by the field of mean angular velocities of the natural particle rotation as well as by the field of mean translational velocities. The state of strain is defined by two strain rate tensors (two strain tensors), and the state of stress by tensors of the ordinary and couple stresses.

Moreover, many important characteristics of the behavior of asymmetric media cannot be determined within the scope of the phenomenological approach. An experimental study also encounters a number of difficulties.

Modern methods of the statistical theory of irreversible processes provide the possibility, in principle, of a theoretical determination of the characteristics of the behavior of the systems under consideration.

Earlier, conservation laws for asymmetric media [5] were given a statistical foundation on the basis of the Liouville equation. Conservation laws and irreversible processes in these media have been examined in [6] by the method of a nonequilibrium statistical operator. The method of correlative functions of conditional distributions [7, 8] has been applied in giving a statistical foundation to the conservation laws and singularities of the kinematics of a medium. The expressions obtained for the stress tensors and the couple stresses afforded a possibility of analyzing existence and symmetry conditions for them.

The present paper is devoted to a construction of a theory of viscosity coefficients and elastic moduli for asymmetric media.

Let us examine a system of N nonspherical molecules characterized by the Hamiltonian

$$H = \frac{1}{2}\sum_{\nu=1}^{N}\left[\frac{(p^{\nu})^2}{m} + \sum_{k=1}^{3}\frac{(s_k'^{\nu})^2}{J_k}\right] + \frac{1}{2}\sum_{\nu,\mu}^{N}\Phi(\mathbf{r}^{\nu\mu}, \boldsymbol{\alpha}^{\nu}, \boldsymbol{\alpha}^{\mu}) \tag{1}$$

$$\boldsymbol{\alpha}^{\nu} = (\alpha_i^{\nu}), \qquad \mathbf{r}^{\nu\mu} = \mathbf{q}^{\mu} - \mathbf{q}^{\nu}$$

Here $\mathbf{p}^{\nu}$ is the particle momentum, canonically conjugate to the radius-vector $\mathbf{q}^{\nu}$ of its center of inertia; $\mathbf{r}^{\nu\mu}$ is the spacing between the particles ν and μ; $\boldsymbol{\alpha}^{\nu}$ is a set of angles defining the particle orientation; $s_k'^{\nu}$ are projections of the spin momentum of the molecule on its principal axes of inertia; J_k are the principal moments of particle inertia; m is its mass; Φ is the potential of pairwise interaction (noncentral forces).

Under the effect of a small mechnical perturbation let the system Hamiltonian change by $\Delta H = -RZ(t)$. Then the mean change in the dynamical variable Q, according to the Kubo theory [9], is defined by the expression

$$\Delta Q = -\frac{1}{\theta}\int_0^{\infty}\langle R(0)\,Q^{\cdot}(s)\rangle\, Z(t-s)\,ds \qquad \left(\theta = kT,\ Q^{\cdot} = \frac{dQ}{dt}\right) \tag{2}$$

Here the symbol $\langle\ \rangle$ denotes averaging over the equilibrium canonical ensemble, k is the Boltzmann constant, T is the absolute temperature. But the definition of ΔH meets known difficulties. A method of determining the increments of the Hamiltonian [10], which is not related to the introduction of additional particular representations on creating a medium flux [11, 12], has been proposed earlier for a system with central and noncentral interaction. A more general approach is proposed below.

Let us examine a system subjected to a small strain whereupon the particles receive the displacement $\mathbf{u}(\mathbf{q})$ and are rotated through the small angle $\varphi(\mathbf{q})$. The change in the coordinate and momentum functions due to the strain can be determined from the relationship for the increments of the functions of the phase coordinates of a system of material points with an infinitesimal canonical transformation [13] generalized to the case of a continuous medium

$$\Delta Q = \int [Qp_j]\, u_j d\mathbf{q} + \int [Ql_j]\, \varphi_j\, d\mathbf{q} \tag{3}$$

The square brackets are here the Poisson brackets, p_j and l_j are the microscopic densities of the momentum and moment of momentum defined by the equalities

$$p_i(\mathbf{q}) = \sum_{\nu=1}^{N} p_i^{\nu}\delta(\mathbf{q}^{\nu} - \mathbf{q}), \qquad l_i(\mathbf{q}) = \sum_{\nu=1}^{N} s_i^{\nu}\delta(\mathbf{q}^{\nu} - \mathbf{q}), \tag{4}$$

in which δ is the delta function; s_i^{ν} the components of the moment of momentum of the particle relative to the frame of reference.

Let us determine the change in the Hamiltonian upon the imposition of a strain by using (3). We find the Poisson brackets from the momentum and moment of momentum conservation laws [6–8]

$$\frac{\partial p_i}{\partial t} = [p_i H] = \frac{\partial \tau_{ij}^{+}}{\partial q_j}, \quad \frac{\partial l_i}{\partial t} = [l_i H] = \frac{\partial \pi_{ij}^{+}}{\partial q_j} + e_{ijk}\tau_{kj}^{+} \tag{5}$$

where e_{ijk} is the Levi-Civita tensor. The flux densities of the momentum (stress tensor) τ_{ik}^+ and the moment of momentum (couple stress tensor) π_{ik}^+ are given by the equalities [6, 8]

$$\tau_{ik}^+ = \sum_{\nu=1}^{N} \left\{ -\frac{p_i^\nu p_k^\nu}{m} + \frac{1}{2} \sum_{\mu\neq\nu}^{N} F_i^{\nu\mu} x_k^{\nu\mu} \right\} \delta(\mathbf{q}^\nu - \mathbf{q}) \tag{6}$$

$$\pi_{ik}^+ = \sum_{\nu=1}^{N} \left\{ -\frac{s_i^\nu p_k^\nu}{m} + \frac{1}{2} \sum_{\mu\neq\nu}^{N} M_i^{\nu\mu} x_k^{\nu\mu} \right\} \delta(\mathbf{q}^\nu - \mathbf{q})$$

Here $x_k^{\nu\mu}$ are components of the radius-vector $\mathbf{r}^{\nu\mu}$, and $F_i^{\nu\mu}$ and $M_i^{\nu\mu}$ are components of the force and moment of the couple exerted by the molecules μ on the molecules ν. Let us note that expressions for the mean stress tensors are presented in [8]. After integrating by parts and neglecting surface integrals, we represent (3) for $Q = H$ as

$$\Delta H = \int \tau_{ik}^+ \varepsilon_{ik} d\mathbf{q} + \int \pi_{ik}^+ \gamma_{ik} d\mathbf{q} \tag{7}$$

Here ε_{ik} and γ_{ik} are the strain tensors

$$\varepsilon_{ik} = \frac{\partial u_i}{\partial q_k} - \varphi_m e_{mki}, \qquad \gamma_{ik} = \frac{\partial \varphi_i}{\partial q_k} \tag{8}$$

Their expressions were earlier given a statistical foundation by another method in a derivation of the energy conservation law [7]. Considering the homogeneous strain case, we write down the final expression for ΔH after integrating and utilizing the properties of the delta function

$$\Delta H = T_{ik}\varepsilon_{ik} + \Pi_{ik}\gamma_{ik} \tag{9}$$

$$T_{ik} = \int \tau_{ik}^+ d\mathbf{q} = -\sum_{\nu=1}^{N} \frac{p_i^\nu p_k^\nu}{m} + \frac{1}{2}\sum_{\nu,\mu}^{N} F_i^{\nu\mu} x_k^{\nu\mu} \tag{10}$$

$$\Pi_{ik} = \int \pi_{ik}^+ d\mathbf{q} = -\sum_{\nu=1}^{N} \frac{s_i^\nu p_k^\nu}{m} + \frac{1}{2}\sum_{\nu,\mu}^{N} M_i^{\nu\mu} x_k^{\nu\mu}$$

According to (9)

$$\Delta H = -R_1 Z_1 - R_2 Z_2, \quad R_1 = -T_{ik}, \quad R_2 = -\Pi_{ik}, \quad Z_1 = \varepsilon_{ik}, \quad Z_2 = \gamma_{ik}$$

Keeping (2) in mind, let us select

$$Q_1 = \frac{1}{V} \int_{-\infty}^{t} (T_{ik} - T_{ik}^\circ)\, dt, \qquad Q_2 = \frac{1}{V} \int_{-\infty}^{t} (\Pi_{ik} - \Pi_{ik}^\circ)\, dt$$

where T_{ik}° and Π_{ik}° are the invariant parts of T_{ik} and Π_{ik} in the sense of [14]. Assuming that the thermodynamic functions of the equilibrium state are independent of the mean densities of the moment of momentum $\langle l \rangle$, by following the proposal in [15] expressions can be written for the invariant parts (the canonical ensemble is utilized) in the form

$$T_{ik}^\circ = \tau_{ik}V + \frac{\partial \tau_{ik}V}{\partial E}(H - E), \qquad \Pi_{ik}^\circ = \pi_{ik}V + \frac{\partial \pi_{ik}V}{\partial E}(H - E) \tag{11}$$

Here τ_{ik}, π_{ik} are the mean equilibrium ordinary and bending stress tensors, E is the internal energy and V the system volume. Then on the basis of (2), in the case of cyclical loading with frequency ω

$$\varepsilon_{ik} = \varepsilon_{ik}(0)\, e^{i\omega t}, \qquad \gamma_{ik} = \gamma_{ik}(0)\, e^{i\omega t}$$

we find that

$$\Delta Q_1 = \left\{ \frac{1}{\theta V} \int_0^\infty e^{-i\omega t} \langle T_{mn}(0) [T_{ik}(t) - T_{ik}^\circ(t)] \rangle \, dt \right\} \varepsilon_{mn} +$$

$$+ \left\{ \frac{1}{\theta V} \int_0^\infty e^{-i\omega t} \langle \Pi_{mn}(0) [T_{ik}(t) - T_{ik}^\circ(t)] \rangle \, dt \right\} \gamma_{mn} \qquad (12)$$

$$\Delta Q_2 = \left\{ \frac{1}{\theta V} \int_0^\infty e^{-i\omega t} \langle T_{mn}(0) [\Pi_{ik}(t) - \Pi_{ik}^\circ(t)] \rangle \, dt \right\} \varepsilon_{mn} +$$

$$+ \left\{ \frac{1}{\theta V} \int_0^\infty e^{-i\omega t} \langle \Pi_{mn}(0) [\Pi_{ik}(t) - \Pi_{ik}^\circ(t)] \rangle \, dt \right\} \gamma_{mn}$$

Comparing these equalities with the phenomenological relationships

$$\int_{-\infty}^{t} \tau_{ik}' dt = a_{ikmn}\varepsilon_{mn} + b_{ikmn}\gamma_{mn} \qquad (13)$$

$$\int_{-\infty}^{t} \pi_{ik}' dt = c_{ikmn}\varepsilon_{mn} + d_{ikmn}\gamma_{mn}$$

we obtain explicit expressions for the tensors a_{ikmn}, b_{ikmn}, c_{ikmn}, d_{ikmn}, of the viscosity coefficients, in the braces in (12). The τ_{ik}', π_{ik}' in (13) are the viscous ordinary and couple stress tensors. It is assumed that $\tau_{ik}' \to 0$, $\pi_{ik}' \to 0$ as $t \to \infty$. The expressions (12) show that the viscosity coefficients possess frequency dispersion, and permit direct investigation of their symmetry properties.

Let the distribution functions used for the averaging in computing the viscosity coefficients be invariant relative to inversion. Then the mean equilibrium couple stress tensor vanishes since it includes the product of the pseudovector M_i by the vector x_k. The expressions for b_{ikmn} and c_{ikmn} are proportional to the combinations $F_i x_k M_m x_n$, therefore, they vanish upon averaging, and the viscosity of an anisotropic medium is characterized by the two tensors a_{ikmn} and d_{ikmn}. An isotropic nongyrotropic medium possesses six viscosity coefficients. In the general case, an isotropic and gyrotropic medium has twelve viscosity coefficients. For systems with central interaction the viscosity is described by the tensor a_{ikmn} whose expression agrees with that obtained earlier in [10].

On the basis of the expressions for the viscosity coefficients, the complex elastic moduli A, B, C, D describing the viscoelastic behavior of a medium under cyclic loading can be obtained

$$A_{ikmn} = \overset{\circ}{A}_{ikmn} + i\omega a_{ikmn}, \qquad B_{ikmn} = \overset{\circ}{B}_{ikmn} + i\omega b_{ikmn} \qquad (14)$$

$$C_{ikmn} = \overset{\circ}{C}_{ikmn} + i\omega c_{ikmn}, \qquad D_{ikmn} = \overset{\circ}{D}_{ikmn} + i\omega d_{ikmn}$$

Here A°, B°, C°, D° are the adiabatic elastic moduli under slow loading $(\omega = 0)$. Passing to the limit $\omega \to \infty$ in (14), the ultimate high-frequency elastic moduli can be obtained for a medium with couple stresses. The passage to the limit was considered in [16, 18] for the isotropic fluid with central interaction of the spherical molecules when the complex volume and shear moduli of elasticity are expressed in terms of the volume η_V and shear η viscosity coefficients by means of the relationships

$$K(\omega) = K_0 + i\omega\eta_V(\omega), \quad \mu(\omega) = i\omega\eta(\omega) \tag{15}$$

The ultimate high-frequency elastic moduli (averaging is performed in the statistical method of conditional distributions in the F_{11} approximation [17] are defined by the equalities

$$K_\infty = \frac{5}{3}\frac{\theta}{v} + \frac{2\pi}{9v^2}\int_{r_0}^{\infty} \frac{d}{dr}\left(\frac{\Phi'(r)}{r^2}\right)\varphi(r)\, r^6\, dr \tag{16}$$

$$\mu_\infty = \frac{\theta}{v} + \frac{2\pi}{15v^2}\int_{r_0}^{\infty} \frac{d}{dr}(r^4\Phi'(r))\,\varphi(r)\, dr \qquad \left(v = \frac{4}{3}\pi r_0^3\right)$$

where v is the molecular volume, r_0 is the radius of the molecular cell, and $v^{-1}\varphi(r)$ represents the conditional distribution function $F_{11}(q_2/q_1)$ [17, 18]. We also obtain expressions for the ultimate high-frequency elastic moduli of an asymmetric medium by another method whose idea is due to Green [19] and which he applied to a system of spherical molecules. To do this we find the change in the linear approximation of the mean stress tensors upon application of a small strain by taking into account the transformation of the space and angular variables, as well as the momentum, in contrast to [19]. The mean stress tensors are determined by the expressions

$$\tau_{ik} = -\frac{1}{m}\iint p_i p_k F_{11}\, d\mathbf{p}\, ds + \frac{1}{2}\iiint F_i x_k F_{11}^{(1)} d\mathbf{r}\, d\boldsymbol{\alpha}\, d\boldsymbol{\alpha}' \tag{17}$$

$$\pi_{ik} = -\frac{1}{m}\iint s_i p_k F_{11}\, d\mathbf{p}\, ds + \frac{1}{2}\iiint M_i x_k F_{11}^{(1)} d\mathbf{r}\, d\boldsymbol{\alpha}\, d\boldsymbol{\alpha}'$$

The averaging is described by using the equilibrium distribution functions $F_{11}(\mathbf{q}, \mathbf{p}, \mathbf{s})$ and $F_{11}^{(1)}(\mathbf{q}, \mathbf{r}, \boldsymbol{\alpha}, \boldsymbol{\alpha}')$. The actual averaging with respect to the momenta will be performed after their transformation. It is convenient to represent the averaging operation as

$$\tau_{ik} = \langle T_{ik}, F\rangle, \qquad \pi_{ik} = \langle \Pi_{ik}, F\rangle \tag{18}$$

Here $\langle\,, F\rangle$ denotes averaging in the sense of [17]. The stress tensor increment in a linear approximation can be written as

$$\Delta\tau_{ik} = \langle \Delta T_{ik}, F\rangle + \langle T_{ik}, \Delta F\rangle \tag{19}$$

$$\Delta\pi_{ik} = \langle \Delta\Pi_{ik}, F\rangle + \langle \Pi_{ik}, \Delta F\rangle$$

The change in the distribution functions as a result of the strain is determined from the condition of invariance of the appropriate probabilities. Since $d\mathbf{q}^+ = d\mathbf{q}\,(1 + \operatorname{div}\mathbf{u})$, then by virtue of the smallness of the strain

$$F_{11}^+ d\mathbf{p}^+ ds^+ = (1 - \operatorname{div}\mathbf{u})\, F_{11}\, d\mathbf{p}\, ds$$

$$F_{11}^{(1)+} d\mathbf{r}^+ d\boldsymbol{\alpha}^+ d\boldsymbol{\alpha}'^+ = (1 - \operatorname{div}\mathbf{u})\, F_{11}^{(1)} d\mathbf{r}\, d\boldsymbol{\alpha}\, d\boldsymbol{\alpha}'$$

This can be written as

$$\Delta F = -F\frac{\partial u_i}{\partial q_i} = -F\delta_{jn}\varepsilon_{jn}$$

Then

$$\Delta\tau_{ik} = \langle \Delta T_{ik} - T_{ik}\delta_{jn}\varepsilon_{jn}, F\rangle, \qquad \Delta\pi_{ik} = \langle \Delta\Pi_{ik} - \Pi_{ik}\delta_{jn}\varepsilon_{jn}, F\rangle \tag{20}$$

Let us determine ΔT_{ik} and $\Delta \Pi_{ik}$ on the basis of (3). Let us apply the relationships

$$\frac{\partial}{\partial \mathbf{q}^{\nu}} = -\frac{\partial}{\partial \mathbf{r}^{\nu\mu}}, \qquad \frac{\partial}{\partial \mathbf{q}^{\mu}} = \frac{\partial}{\partial \mathbf{r}^{\nu\mu}} \tag{21}$$

$$\delta(\mathbf{q}^{\nu} - \mathbf{q}) - \delta(\mathbf{q}^{\mu} - \mathbf{q}) = x_n^{\nu\mu} \frac{\partial \delta(\mathbf{q}^{\nu} - \mathbf{q})}{\partial q_n} + \ldots \tag{22}$$

to transform the appropriate Poisson brackets. To evaluate the Poisson brackets containing the density of the moment of momentum, we moreover utilize

$$\frac{\partial Q}{\partial \varphi_j^{\nu}} + \frac{\partial Q}{\partial \varphi_j^{\mu}} + e_{jnm} x_n^{\nu\mu} \frac{\partial Q}{\partial x_m^{\nu\mu}} = 0 \tag{23}$$

It expresses the invariance of the function $Q(\mathbf{r}^{\nu\mu}, \boldsymbol{\alpha}^{\nu}, \boldsymbol{\alpha}^{\mu})$ of the variables of two particles relative to transformations conserving the mutual orientation of the particles and the radius-vector connecting their centers of inertia. The quantities $\partial Q / \partial \varphi_i$ denote proportionality coefficients between the linear part of the increment in the function Q (because of the change in the angular variables of one particle) and the components of the angle of slight rotation φ_i. After having evaluated the quantities ΔT_{ik} and $\Delta \Pi_{ik}$ we write the average of the relations (20) explicitly, and averaging with respect to the momenta, we obtain

$$\Delta \tau_{ik} = \Big[F_{11}\theta(\delta_{ij}\delta_{kn} + \delta_{kj}\delta_{in}) - \tau_{ik}\delta_{jn} + \tag{24}$$

$$+ \frac{1}{2}\iiint \frac{\partial F_i x_k}{\partial x_j} x_n F_{11}^{(1)}\, d\mathbf{r}\, d\boldsymbol{\alpha}\, d\boldsymbol{\alpha}'\Big] \varepsilon_{jn} + \left[\frac{1}{2}\iiint \frac{\partial M_i}{\partial x_j} x_k x_n F_{11}^{(1)}\, d\mathbf{r}\, d\boldsymbol{\alpha}\, d\boldsymbol{\alpha}'\right] \gamma_{jn}$$

$$\Delta \pi_{ik} = \left[-\pi_{ik}\delta_{jn} + \frac{1}{2}\iiint \frac{\partial M_i x_k}{\partial x_j} x_n F_{11}^{(1)}\, d\mathbf{r}\, d\boldsymbol{\alpha}\, d\boldsymbol{\alpha}'\right] \varepsilon_{jn} +$$

$$+ \left[\frac{F_{11}\theta}{m}\left(\sum_{k=1}^{3} I_k\right)\delta_{kn}\delta_{ij} + \frac{1}{2}\iiint \frac{\partial M_i}{\partial \varphi_j'} x_k x_n F_{11}^{(1)}\, d\mathbf{r}\, d\boldsymbol{\alpha}\, d\boldsymbol{\alpha}'\right] \gamma_{jn}$$

Here M_i is the moment of the couple acting on the molecule whose orientation is determined by the set of angles $\boldsymbol{\alpha}$. The expressions in the square brackets define the tensors of the elastic moduli A^{∞}, B^{∞}, C^{∞}, D^{∞} of the asymmetric medium under high-frequency loading

$$\Delta \tau_{ik} = A_{ikmn}^{\infty}\varepsilon_{mn} + B_{ikmn}^{\infty}\gamma_{mn}, \quad \Delta \pi_{ik} = C_{ikmn}^{\infty}\varepsilon_{mn} + D_{ikmn}^{\infty}\gamma_{mn}$$

We see that the elastic moduli now under consideration are ultimate for $\omega \to \infty$ by the example of an isotropic medium with central interaction, which is characterized by the one elastic modulus tensor A_{ikmn}^{∞}. Taking into account that $F_i = r^{-1}\Phi'(r) x_i$, we find after averaging according to the orientations that

$$A_{ikmn}^{\infty} = \left[\frac{\theta}{v} - \frac{2\pi}{3v^2}\int_{r_0}^{\infty} \Phi'(r)\,\varphi r^3\, dr + \frac{2\pi}{15v^2}\int_{r_0}^{\infty} \frac{d}{dr}\left(\frac{\Phi'(r)}{r}\right)\varphi r^5\, dr\right]\delta_{ik}\delta_{mn} +$$

$$+ \left[\frac{\theta}{v} + \frac{2\pi}{15v^2}\int_{r_0}^{\infty} \frac{d}{dr}\left(r^4\Phi'(r)\right)\varphi\, dr\right](\delta_{im}\delta_{kn} + \delta_{in}\delta_{km})$$

It hence follows that the expressions for K_{∞} and μ_{∞} agree completely with the expressions (16) obtained by using the passage to the limit in the formulas for the complex elastic moduli.

The reasoning on the symmetry of the elastic moduli tensors is analogous to that utilized earlier for the viscosity coefficient tensors and leads to similar results. It can here still be established easily that in the absence of initial stresses the tensors A and D are symmetric relative to commutation of the first and second pair of subscripts, and $B = C$, because of the equalities

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}, \qquad \frac{\partial M_i}{\partial \varphi_j'} = \frac{\partial M_j}{\partial \varphi_i'}$$

Let us obtain an expression for the adiabatic elastic moduli of asymmetric fluids by again considering a change in the mean stress tensors upon application of a slight strain. Averaging with respect to the momenta is here carried out in the kinetic parts of the stress tensors by using the Maxwell distribution

$$\tau_{ik} = -F_{11}\theta\delta_{ik} + \frac{1}{2}\iiint F_i x_k F_{11}^{(1)}\, d\mathbf{r}\, d\boldsymbol{\alpha}\, d\boldsymbol{\alpha}' \tag{25}$$

$$\pi_{ik} = \frac{1}{2}\iiint M_i x_k F_{11}^{(1)}\, d\mathbf{r}\, d\boldsymbol{\alpha}\, d\boldsymbol{\alpha}'$$

For fluids under adiabatic loading ($\omega = 0$) the increment in the mean stress tensors is determined by the change in the temperature and volume

$$\Delta\tau_{ik} = \left(\frac{\partial\tau_{ik}}{\partial T}\right)_v \Delta T + \left(\frac{\partial\tau_{ik}}{\partial v}\right)_T v\delta_{jn}\varepsilon_{jn} \tag{26}$$

$$\Delta\pi_{ik} = \left(\frac{\partial\pi_{ik}}{\partial T}\right)_v \Delta T + \left(\frac{\partial\pi_{ik}}{\partial v}\right)_T v\delta_{jn}\varepsilon_{jn}$$

The change in medium temperature under adiabatic strain is expressed by a formula generalizing the known relationship for an elastic body with a symmetric stress tensor [20]

$$\Delta T = \frac{Tv}{C_v}\left[\left(\frac{\partial\tau_{ik}}{\partial T}\right)_v \varepsilon_{ik} + \left(\frac{\partial\pi_{ik}}{\partial T}\right)_v \gamma_{ik}\right] \tag{27}$$

where C_v is the specific heat at constant volume referred to one particle. The generalization includes the appearance of the second member.

On the basis of (26) and (27), expressions are also obtained for the tensors of the adiabatic elastic moduli

$$\mathring{A}_{ikmn} = \frac{Tv}{C_v}\left(\frac{\partial\tau_{ik}}{\partial T}\right)_v\left(\frac{\partial\tau_{mn}}{\partial T}\right)_v + \left(\frac{\partial\tau_{ik}}{\partial v}\right)_T v\delta_{mn}, \quad \mathring{B}_{ikmn} = \frac{Tv}{C_v}\left(\frac{\partial\tau_{ik}}{\partial T}\right)_v\left(\frac{\partial\pi_{mn}}{\partial T}\right)_v$$

$$\mathring{C}_{ikmn} = \frac{Tv}{C_v}\left(\frac{\partial\pi_{ik}}{\partial T}\right)_v\left(\frac{\partial\tau_{mn}}{\partial T}\right)_v + \left(\frac{\partial\pi_{ik}}{\partial v}\right)_T v\delta_{mn}, \quad \mathring{D}_{ikmn} = \frac{Tv}{C_v}\left(\frac{\partial\pi_{ik}}{\partial T}\right)_v\left(\frac{\partial\pi_{mn}}{\partial T}\right)_v \tag{28}$$

The passage to a fluid with central interaction taking into account that $\tau_{ik} = -P\delta_{ik}$ (P is the pressure) results in the known formula

$$\mathring{A}_{ikmn} = K_0\delta_{ik}\delta_{mn}, \qquad K_0 = \frac{Tv}{C_v}\left(\frac{\partial P}{\partial T}\right)_v^2 - v\left(\frac{\partial P}{\partial v}\right)_T$$

The expressions obtained for the viscosity coefficients and elastic moduli contain a dependence on the parameters of intramolecular interaction and the thermodynamic state of the medium, and permit evaluation. The expressions for the high-frequency and adiabatic elastic moduli are also applicable to the determination of the viscosity coefficients involving the appropriate relaxation times [21].

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CLASSIFICATION OF LINEAR INTEGRALS OF A HOLONOMIC MECHANICAL SYSTEM WITH TWO DEGREES OF FREEDOM

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The linear integrals of a mechanical system are classified according to the solutions of the Killing equation and of the form of the generalized forces. An example of a mechanical system with two degrees of freedom which has a generalized force function but no linear integral, was given in [1].

Let $\lambda_\varkappa \dot{q}^\varkappa = c$ be the linear integral of a mechanical system with two degrees of freedom. This requires that [1]

$$\nabla_s \lambda_\varkappa + \nabla_\varkappa \lambda_s = 0 \tag{1}$$

$$\lambda_\varkappa Q^\varkappa = 0 \tag{2}$$

Considering

$$2T dt^2 = ds^2 = g_{\lambda\mu} dq^\lambda dq^\mu$$

as a linear element of the two-dimensional Riemannian space V_2 we find, that the following possibilities [2, 3] may be given to the Killing equation (1). Equations (1) have:

a) no solution,

b) one solution, or

c) three solutions.

In the case (a) Eq. (1) has no solution, hence the mechanical system has no linear integral. In the cases (b) and (c), using integrable transformations the linear element can be reduced to the form

$$2T\ dt^2 = ds^2 = V(q^1)\ [(dq^1)^2 + (dq^2)^2] \tag{3}$$

then it is said that the rotation metric is given [2]. It has been shown that each rotation